THE DEFORMATION QUANTIZATION IN THE CONTEXT OF KONTSEVICH

FRANK KLINKER

Abstract. We describe the quantization procedure for Poisson structures as proposed by Kontsevich in [6].

1. On the tensor product of graded spaces: The décalage isomorphism

Let $V_j = \bigoplus_{i\in\mathbb{Z}} V_j^{(i)}$ be $\mathbb{Z}$-graded vector spaces for $j = 1, \ldots, n$ and denote by $V_j[k]$ the $\mathbb{Z}$-graded vector spaces obtained by the shift by k, i.e. $V_j[k]^{(i)} := V_j^{(i+k)}$. This may be rephrased in

$$V_j[k] = V_j \otimes \mathbb{R}[k] \quad \text{with} \quad \mathbb{R}[k]^{(l)} = \begin{cases} \mathbb{R} & l = -k \\ 0 & \text{else} \end{cases} . \tag{1}$$

Consider the following isomorphism

$$\begin{aligned} &V_1[k_1] \otimes \cdots \otimes V_n[k_n] \simeq \big(V_1 \otimes \cdots \otimes V_n\big)[k_1 + \cdots + k_n] \\ &v_1^{(\alpha_1)} \otimes \ldots \otimes v_n^{(\alpha_n)} \mapsto (-)^{\alpha_2 k_1 + \alpha_3(k_1+k_2)+\cdots+\alpha_n(k_1+\cdots+k_{n-1})} v_1^{(\alpha_1)} \otimes \ldots \otimes v_n^{(\alpha_n)}. \end{aligned} \tag{2}$$

The sign in this formula can be formally constructed by rearranging the following tuple

$$(\alpha_1, k_1; \alpha_2, k_2; \ldots; \alpha_{n-1}, k_{n-1}; \alpha_n, k_n) \to (\alpha_1, \cdots, \alpha_n; k_1, \cdots, k_n) .$$

In particular, $V_j = V$ and $k_j = 1$ gives

$$\begin{aligned} V[1]^{\otimes n} &\simeq V^{\otimes n}[n] \\ v_1^{(\alpha_1)} \otimes \ldots \otimes v_n^{(\alpha_n)} &\mapsto (-)^{\sum_{j=1}^{n}(j-1)\alpha_j} v_1^{(\alpha_1)} \otimes \ldots \otimes v_n^{(\alpha_n)} \\ &= (-)^{\alpha_2+\alpha_4+\cdots+\alpha_{2[\frac{n}{2}]}} v_1^{(\alpha_1)} \otimes \ldots \otimes v_n^{(\alpha_n)} \end{aligned} \tag{3}$$

This has the following consequence on the symmetrized resp. anti-symmetrized tensor products

$$S^n(V[1]) \simeq (\Lambda^n V)[n] , \tag{4}$$

Email: mail@frank-klinker.de.

This text is based on a talk presented at a joint seminar of the groups *Differential Geometry* and *Theoretical Physics III* at TU Dortmund.

To prove this we have to show the compatibility of the product structures with (2):

$$v_1^{(\alpha_1)} \vee \ldots \vee v_n^{(\alpha_n)} = (-)^{(\alpha_i - 1)\sum_{j=1}^{i-1}(\alpha_j - 1)} v_i^{(\alpha_i)} \vee v_1^{(\alpha_1)} \vee \ldots \vee \widehat{v_i^{(\alpha_i)}} \vee \ldots \vee v_n^{(\alpha_n)}$$

$$\Big\downarrow (2)$$

$$(-)^{\sum_{j=1}^{n}(j-1)\alpha_j} v_1^{(\alpha_1)} \wedge \ldots \wedge v_n^{(\alpha_n)} \;=$$

$$= (-)^{\sum_{j=1}^{n}(j-1)\alpha_j} (-)^{i-1} (-)^{\alpha_i \sum_{j=1}^{i-1}\alpha_j} v_i^{(\alpha_i)} \wedge v_1^{(\alpha_1)} \wedge \ldots \wedge \widehat{v_i^{(\alpha_i)}} \wedge \ldots \wedge v_n^{(\alpha_n)}$$

The sign $(-)^{\sum_{j=1}^{i-1} j\alpha_j + \sum_{j=i+1}^{n}(j-1)\alpha_j}$ arises when we compare the two right hand sides due to (2). This is the proof of (4) because

$$\sum_{j=1}^{n}(j-1)\alpha_j + (i-1) + \alpha_i \sum_{j=1}^{i-1}\alpha_j$$
$$\equiv (\alpha_i - 1)\sum_{j=1}^{i-1}(\alpha_j - 1) + \sum_{j=1}^{i-1} j\alpha_j + \sum_{j=i+1}^{n}(j-1)\alpha_j \mod 2\,. \quad (5)$$

2. Co-Algebras, DGLAs, L_∞-algebras and associated structures

A differential graded Lie algebra (DGLA) is a triple $(\mathfrak{g}, [\cdot,\cdot], d)$ consisting of a $\mathbb{Z}$-graded vector space $\mathfrak{g}$, a graded anti-symmetric bilinear map $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ of degree zero and a linear map $d : \mathfrak{g} \to \mathfrak{g}$ of degree one. The bracket obeys the graded Jacobi identity

$$(-)^{zx}[x,[y,z]] + (-)zy[z,[x,y]] + (-)^{xy}[y,[z,x]] = 0 \quad (6)$$

and the map d is a differential, i.e. $d^2 = 0$ and it is compatible with the bracket:

$$d[x,y] = [dx,y] + (-)^x[x,dy]\,. \quad (7)$$

The Jacobi identity (6) can be reformulated as $ad_x([y,z]) = [ad_x(y),z] + (-)^{xy}[y, ad_x(z)]$. In this way ad_x is a derivation of degree $|x|$.

A $\mathbb{Z}$-graded co-algebra (without unit) is a $\mathbb{Z}$-graded vector space $\mathcal{A} = \bigoplus_{k\in\mathbb{Z}} \mathcal{A}^{(k)}$ with a co-multiplication $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ of degree zero which is associative in the following sense:

$$\begin{array}{ccc} \mathcal{A}\otimes\mathcal{A} & \xrightarrow{\Delta\otimes \mathrm{id}} & \mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \\ \Delta\uparrow & \circlearrowleft & \uparrow \mathrm{id}\otimes\Delta \\ \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A}\otimes\mathcal{A} \end{array}$$

For example the tensor algebra $T(V) = \bigoplus_{n\geq 0} V^{\otimes n}$ of a $\mathbb{Z}$-graded vector space $V = \bigoplus_{k\in\mathbb{Z}} V^{(k)}$ is a co-algebra with

$$\Delta(v_1 \otimes \cdots \otimes v_n) = \\ = 1\otimes(v_1\otimes\cdots\otimes v_n)+\sum_{r=1}^{n-1}(v_1\otimes\ldots\otimes v_r)\otimes(v_{r+1}\otimes\cdots\otimes v_n)+(v_1\otimes\cdots\otimes v_n)\otimes 1. \tag{8}$$

This induces a of co-algebra structure on $S(V) = T(V)/\{v\otimes w-(-)^{vw}w\otimes v\}$, too.

A DIFFERENTIAL on a co-algebra is a linear map Q with $Q^2 = 0$ which is compatible with the co-multiplication in the following sense. For $\Delta(x) = \sum x_i^1 \otimes x_i^2$ we have

$$\Delta \circ Q(x) = (Q \otimes \mathrm{id}) \circ \Delta(x) + \sum (-)^{x_i^1}(\mathrm{id} \otimes Q)(x_i^1 \otimes x_i^2). \tag{9}$$

An L_∞-ALGEBRA is pair $(\mathfrak{g}, Q)$ where $\mathfrak{g}$ is a $\mathbb{Z}$-graded vector space and Q is a graded co-algebra differential on $S(\mathfrak{g}[1])$.

The differential Q can be uniquely described by its Taylor coefficients $Q_k : S^k(\mathfrak{g}[1]) \to \mathfrak{g}[1]$ of degree 1. They are given by

$$Q(v_1 \vee \cdots \vee v_n) = \\ = \sum_{k=1}^{n} \sum_{\sigma\in\mathcal{S}_n} \frac{\binom{n}{k}}{n!}\epsilon(v_{\sigma(1)}, \ldots, v_{\sigma(n)})Q_k(v_{\sigma(1)}\vee\cdots\vee v_{\sigma(k)})\vee v_{\sigma(k+1)}\vee\cdots\vee v_{\sigma(n)}, \tag{10}$$

where $\epsilon(v_{i_1}, \ldots, v_{i_n})$ is the sign which we obtain when we bring back the vectors in the right order.

The condition $Q^2 = 0$ translates to the coefficients into a series of equations of which the first three are given by[1]

$$0 = Q_1^2(v)\,, \tag{11}$$
$$0 = Q_1 \circ Q_2(v \vee w) + Q_2(Q_1(v) \vee w) + (-)^v Q_2(v \vee Q_1(w))\,, \tag{12}$$

and

$$\begin{aligned} Q_2(Q_2(v\vee w)\vee z)+(-)^{zw}Q_2(Q_2(v\vee z)\vee w)+(-)^{v(w+z)}Q_2(Q_2(w\vee z)\vee v) \\ = Q_1 \circ Q_3(v \vee w \vee z) + Q_3(Q_1(v) \vee w \vee z) + (-)^{vw}Q_3(Q_1(w) \vee v \vee z) \\ + (-)^{z(v+w)}Q_3(Q_1(z) \vee v \vee w)\,. \end{aligned} \tag{13}$$

Because of the décalage isomorphism (4) the Q_k can be seen as degree zero maps

$$Q_k : \Lambda^k(\mathfrak{g}) \to \mathfrak{g}[2 - n]\,. \tag{14}$$

[1]The degrees are taken in $\mathfrak{g}[1]$.

An L_∞-MORPHISM between two L_∞-algebras $(\mathfrak{g}, Q^{\mathfrak{g}})$ and $(\mathfrak{h}, Q^{\mathfrak{h}})$ is a degree zero co-algebra morphism $U : S(\mathfrak{g}[1]) \to S(\mathfrak{h}[1])$ compatible with the differential, i.e.

$$U \circ Q^{\mathfrak{g}} \;=\; Q^{\mathfrak{h}} \circ U\,. \tag{15}$$

As in the case of the differential, an L_∞-morphism is uniquely described by its Taylor coefficients $U_k : S^k(\mathfrak{g}[1]) \to \mathfrak{h}[1]$ of degree zero. They are explicitly given by

$$\begin{aligned} U(v_1 \vee \cdots \vee v_n) &= \sum_{r\geq 1} \frac{1}{r!} \sum_{\substack{I_1\cup\cdots\cup I_r=\{1,\ldots n\}\\ k_j:=|I_j|\geq 1}} \epsilon(v_{i_1}\ldots, v_{i_n})\cdot \\ &\qquad \cdot U_{k_1}(v_{i_1}\vee\cdots\vee v_{i_{k_1}})\vee\cdots\vee U_{k_r}(v_{i_{k-k_r+1}}\vee\cdots\vee v_{i_n}) \\ &= \sum_{r\geq 1}\frac{1}{r!}\sum_{k_1+\cdots+k_r=n}\frac{1}{k_1!\cdots k_r!}\sum_{\sigma\in\mathcal{S}_n}\epsilon(v_{\sigma(1)},\ldots,v_{\sigma(n)})\cdot \\ &\qquad \cdot U_{k_1}(v_{\sigma(1)}\vee\cdots\vee v_{\sigma(k_1)})\vee\cdots\vee U_{k_r}(v_{\sigma(k-k_r+1)}\vee\cdots\vee v_{\sigma(n)})\,. \end{aligned}$$

The compatibility condition (15) written in the Taylor coefficients of Q and U gives an infinite series of equations of which the first two are given by

$$\begin{aligned} Q_1^{\mathfrak{h}} \circ U_1(v) \;&=\; U_1 \circ Q_1^{\mathfrak{g}}(v)\,, \\ Q_1^{\mathfrak{h}} \circ U_2(v\vee w) - U_2(Q_1^{\mathfrak{g}}(v)\vee w) - (-)^v U_2(v \wedge Q_1^{\mathfrak{h}}(w)) \;&=\; \\ =\; U_1 \circ Q_2^{\mathfrak{g}}(v\wedge w) - Q_2^{\mathfrak{h}}(U_1(v)\wedge U_1(w))\,. \end{aligned} \tag{16}$$

As before we can treat the Taylor coefficients as maps

$$U_k : \Lambda^k(\mathfrak{g}) \to \mathfrak{h}[1-n] \tag{17}$$

due to (4).

We see that the first Taylor coefficient of U gives rise to a morphism of complexes

$$U_1 : (\mathfrak{g}, Q_1^{\mathfrak{g}}) \to (\mathfrak{h}, Q_1^{\mathfrak{h}})\,. \tag{18}$$

and so induces a morphism on cohomology level. An L_∞-morphism U is called QUASI-ISOMORPHISM if U_1 is an isomorphism on the level of cohomologies. An L_∞-algebra is called FORMAL if it is quasi-isomorphic to its cohomology (with respect to Q_1) considered as L_∞-algebra with induced Q_2 and $Q_i = 0$ for $i \neq 2$, i.e.

$$\big(\mathfrak{g}, Q = (Q_1, Q_2, \ldots)\big) \;\overset{\text{q.i.}}{\simeq}\; \big(H^*(\mathfrak{g}, Q_1), Q = (0, (Q_2)_{\text{ind}}, 0, \ldots)\big)\,.$$

Quasi-isomorphism is an equivalence relation, i.e. for two L_∞-algebras and a quasi-isomorphism $U : (\mathfrak{g}, Q^{\mathfrak{g}}) \to (\mathfrak{h}, Q^{\mathfrak{h}})$ there exists an extension of the map $\widetilde{U}_1 := U_1^{-1}$ on cohomology level to an L_∞-morphism $\widetilde{U} : (\mathfrak{h}, Q^{\mathfrak{h}}) \to (\mathfrak{g}, Q^{\mathfrak{g}})$.

3. The basic examples of L_∞-algebras

Every DGLA gives rise to an L_∞-algebra via the definitions

$$Q_1 := d, \quad Q_2 := [\cdot,\cdot], \quad Q_k := 0 \text{ for } k \geq 3\,. \tag{19}$$

To be more precise, we have to take into account the décalage when changing from $\mathfrak{g}$ to $\mathfrak{g}[1]$. The exact translation for the structures is

$$Q_1(v) = (-)^v dv \quad \text{and} \quad Q_2(v \vee w) = (-)^{v(w-1)}[v,w] \tag{20}$$

where the degrees have been taken in $\mathfrak{g}$.

The conditions (11)-(13 are translated in the following way. We have

$$\begin{aligned} Q_1(Q_2(v\vee w)) &= (-)^{v(w-1)}(-)^{v+w}d[v,w] = (-)^{vw+w}d[v,w]\,,\\ Q_2(Q_1(v)\vee w) &= (-)^v(-)^{(v+1)(w-1)}[dv,w] = -(-)^{vw+w}[dv,w]\,,\\ (-)^{v+1}Q_2(v\vee Q_1(w)) &= (-)^{v+1}(-)^w(-)^{vw}[v,dw] = -(-)^v(-)^{vw+w}[v,dw]\,. \end{aligned}$$

Adding these three lines after multiplying by $(-)^{w+vw}$ yields

$$d[v,w] - [dv,w] - (-)^v[v,dw] \;=\; 0 \tag{21}$$

which is exactly (7). Furthermore,

$$\begin{aligned} Q_2(Q_2(v\vee w)\vee z) &= (-)^{v(w-1)}(-)^{(v+w)(z-1)}[[v,w],z]\\ &= (-)^{vw+wz+w}(-)^{vz}[[v,w],z]\,,\\ (-)^{(z+1)(w+1)}Q_2(Q_2(v\vee z)\vee w) &= (-)^{(z+1)(w+1)}(-)^{vw+vz+wz}(-)^z[[v,z],w]\\ &= (-)^{vw+wz+w}(-)^{zw}[[z,v],w]\,,\\ (-)^{(v+1)(w+z)}Q_2(Q_2(w\vee z)\vee v) &= (-)^{(v+1)(w+z)}(-)^{vw+vz+wz}(-)^z[[w,z],v]\\ &= (-)^{vw+wz+w}(-)^{vw}[[w,z],v]\,, \end{aligned}$$

which yields (6) after adding and dividing out the common factor, i.e.

$$(-)^{vz}[[v,w],z] + (-)^{zw}[[z,v],w] + (-)^{vw}[[w,z],v] = 0\,. \tag{22}$$

In terms of two DGLAs the compatibility condition for the L_∞ morphism is given by an infinite series of equations:

$$\begin{aligned} d\circ U_1(v) \;&=\; U_1\circ d(v)\,,\\ dU_2(v\wedge w) - U_2(dv\wedge w) - (-)^v U_2(v\wedge dw) \;&=\\ =\; U_1([v,w]) - [U_1(v),U_1(w)]\,,&\\ \vdots\quad& \end{aligned} \tag{23}$$

4. The Maurer-Cartan map on L_∞-algebras

Consider an L_∞-algebra $(\mathfrak{g}, Q)$. When we extend the differential $\hbar$-linear $(\mathfrak{g}[[\hbar]], Q)$ turns into an L_∞-algebra, too. We define the generalized Maurer-Cartan map by

$$\begin{aligned} &MC : \hbar\mathfrak{g}^{(1)}[[\hbar]] \to \hbar\mathfrak{g}[[\hbar]] \\ &MC(\hbar x) = \mathrm{proj}_{\mathfrak{g}[[\hbar]]} \circ Q \circ \exp(\hbar x) \end{aligned} \tag{24}$$

where we defined $Q_0 = 0$ and $\exp : \hbar\mathfrak{g}^{(1)}[[\hbar]] \to S(\mathfrak{g}[1])[[\hbar]])$ in the obvious way. We have $Q(\overbrace{x \vee \cdots \vee x}^{k \text{ times}}) = \sum_{j=1}^{k} \binom{k}{j} Q_j(x^j) x^{k-j}$ because $|x| = 0$ in $\mathfrak{g}[1]$ which yields

$$\begin{aligned} MC(\hbar x) = &\ \mathrm{proj}_{\mathfrak{g}[[\hbar]]}\big(\sum_k \frac{\hbar^k}{k!} \sum_{j=1}^{k} \binom{k}{j} Q_j(x^j) x^{k-j}\big) \\ = &\ \mathrm{proj}_{\mathfrak{g}[[\hbar]]}\big(\sum_r \frac{\hbar^r}{r!} x^r \sum_s \frac{\hbar^s}{s!} Q_s(x^s)\big) \ = \ \sum_s \frac{\hbar^s}{s!} Q_s(v^s)\,. \end{aligned} \tag{25}$$

The first terms in this series are given by $\hbar Q_1(x) + \frac{\hbar^2}{2} Q_2(x, x) + o(\hbar^3)$. In the case of a DGLA with $Q_1 = -d$, $Q_2 = [\cdot, \cdot]$ and $Q_{\geq 3} = 0$ we recover – for $\hbar = 1$ – the usual Maurer-Cartan map evaluated on $-x \in \mathfrak{g}^{(1)}$.

We denote the zero set of the Maurer-Cartan map (modulo an action of a gauge group) by $MC(\mathfrak{g})$. It is invariant under L_∞-morphisms and a quasi-isomorphism U provides a bijection, given by

$$MC(\mathfrak{g}) \ni \hbar v \mapsto \sum_m \frac{\hbar^m}{m!} U_m(x^m) = \hbar\widetilde{v} \in MC(\mathfrak{h})\,. \tag{26}$$

We have

$$\begin{aligned} U(exp(\hbar x)) = U(\sum \frac{\hbar^n}{n!}(x^n)) \ &= \ \sum_n \frac{\hbar^n}{n!} U(x^n) \\ &= \sum_n \frac{\hbar^n}{n!} \sum_{r\geq 1} \frac{1}{r!} \sum_{k_1+\cdots+k_r=n} \frac{n!}{k_1! \cdots k_r!} U_{k_1}(v^{k_1}) \vee \cdots \vee U_{k_r}(v^{k_r}) \\ &= \sum_r \frac{1}{r!} \sum_{k_1} \frac{\hbar^{k_1}}{k_1!} U_{k_1}(v^{k_1}) \cdots \sum_{k_r} \frac{\hbar^{k_r}}{k_r!} U_{k_r}(v^{k_r}) \\ &= \sum_r \frac{1}{r!} \big(\sum_m \frac{\hbar^m}{m!} U_m(v^m)\big)^r \ = \ \exp(\hbar\widetilde{v})\,. \end{aligned}$$

This element is in $MC(\mathfrak{h})$, because

$$Q(exp(\hbar\widetilde{v}) = Q \circ U(exp(\hbar x)) = U \circ Q(exp(\hbar x)) = 0\,. \tag{27}$$

5. Kontsevitch's formality

We consider the following two DGLAs.

• The DGLA of poly-vector fields on the manifold M:

$(\mathfrak{T}_{\text{poly}}, [\cdot,\cdot]_{\text{S}}, d=0)$ with $\mathfrak{T}_{\text{poly}}(M) = \mathfrak{X}(M)[1]$ and
$mathfrakX(M)^{(k)} = \Gamma(\Lambda^k TM)$ and $[\cdot,\cdot]_{\text{S}}$ is the Schouten bracket which extends the Lie bracket on vector fields to all poly-vector fields.

A Poisson structure on M is given by a bi-vector field $\pi \in \mathfrak{T}^{(1)}_{\text{poly}}(M)$ obeying $[\pi,\pi] = 0$. We have $Q(\exp(\hbar\pi)) = \frac{\hbar^2}{2}[\pi,\pi]$ for $\hbar\pi \in \hbar\mathfrak{T}^{(1)}_{\text{poly}}(M) \subset \hbar\mathfrak{T}^{(1)}_{\text{poly}}(M)[[\hbar]]$. This allows the following identification

$$\big\{\text{Poisson structures on } M\big\} \subset MC(\mathfrak{T}_{\text{poly}}(M)) \tag{28}$$

• The DGLA of poly-differential operators on M:

$(\mathfrak{D}_{\text{poly}}(M), [\cdot,\cdot]_{\text{G}}, d_{\text{H}})$ with $\Phi \in \mathfrak{D}^{(k)}_{\text{poly}}(M) \subset \text{Hom}(C^\infty(M)^{\otimes(k+1)}, C^\infty(M))$, if Φ is a derivation in every entry. The bracket is the Gerstenhaber bracket defined by

$$\begin{aligned}
&[\Phi^{(k)}, \Psi^{(l)}]_{\text{G}}(a_1,\ldots,a_{k+l-1}) = \\
&\quad = \sum_{j=1}^{k} (-)^{(j-1)(l-1)} \Phi(a_1,\ldots,a_{j-1}, \Psi(a_j,\ldots,a_{j+l-1}), a_{j+l},\ldots,a_{k+l-1}) \\
&\quad - \sum_{j=1}^{l} (-)^{(l+j)(k-1)} \Psi(a_1,\ldots,a_{j-1}, \Phi(a_j,\ldots,a_{j+k-1}), a_{j+k},\ldots,a_{k+l-1})
\end{aligned}$$

(here the degrees are taken in the Hochschild complex). The differential is the one of the Hochschild complex, i.e.

$$\begin{aligned}
&d_{\text{H}}\Phi(f_1\otimes\cdots\otimes f_{k+1}) = f_1\Phi(f_2\otimes\cdots\otimes f_{k+1}) + \\
&+ \sum_{i=1}^{k} (-)^i \Phi(f_1\otimes\cdots\otimes f_i f_{i+1}\otimes\cdots\otimes f_{k+1}) + (-)^{k+1}\Phi(f_1\otimes\cdots\otimes f_k) f_{k+1}.
\end{aligned}$$

The Maurer-Cartan set for the poly-differential operators is given by the $*$-products on M, i.e.

$$\big\{ *\text{-products on } M\big\} = MC(\mathfrak{D}_{\text{poly}}(M)) \tag{29}$$

The Hochschild-Kostant-Rosenberg Theorem says that the natural map

$$\widehat{U}_1 : \mathfrak{T}_{\text{poly}}(M) \to \ \mathfrak{D}_{\text{poly}}(M)\,, \tag{30}$$

that is given by[2]

$$\widehat{U}_1(h)(1) = h$$

for $k = -1$ and

$$\widehat{U}_1(v_1 \wedge \cdots \wedge v_{k+1})(f_1 \otimes \cdots \otimes f_{k+1}) = \frac{1}{(k+1)!} \sum_{\sigma \in \mathcal{S}_{k+1}} (-)^\sigma \prod_{j=1}^{k+1} v_{\sigma(j)}(f_j)$$

for $k \geq 0$, is a quasi-isomorphism of complexes. This map fails to be compatible with the brackets on the two DGLAs. This will be repaired by turning to the L_∞-structures and the construction of an L_∞-morphism U with first Taylor coefficient $U_1 = \widehat{U}_1$. Then U is an quasi isomorphism. We have the following theorem for the special case $M = \mathbb{R}^D$

Theorem 5.1 (Kontsevitch's formality theorem)**.** *The L_∞-algebra obtained from the DGLA $(\mathfrak{D}_{\mathrm{poly}}(\mathbb{R}^D), [\cdot,\cdot], d)$ of poly-differential operators on $\mathbb{R}^D$ is formal.*

Corollary 5.2. Every Poisson structure on $\mathbb{R}^D$ is quantizable.

From now on we restrict ourselves to this special case!

6. Admissible graphs and associated maps

We consider GRAPHS as finite subsets of $\mathbb{C}$ consisting of vertices, which are points in $\{\mathrm{Im} z \geq 0\}$, and edges, i.e. lines connecting two vertices. We distinguish two kinds of vertices and we label them by non-bared and bared integers. The edges are only allowed to start at one kind of vertex. Furthermore they are not allowed to end at the same vertex from which they start. We list these restrictions and further definitions related with graphs

(A) Vertices:

$$v \in \mathcal{V} := \mathcal{V}_1 \dot\cup \mathcal{V}_2 = \{1, \ldots, n\} \dot\cup \{\bar{1}, \ldots, \bar{m}\},$$

$n, m \in \mathbb{Z}$. We call non-bared elements vertices of first kind and bared ones vertices of second kind.

(B) Edges:

$$e \in \mathcal{E} := \{1, \ldots, n\} \times \left(\{1, \ldots, n\} \dot\cup \{\bar{1}, \ldots, \bar{m}\}\right) \setminus \{(v,v); v \in \mathcal{V}_1\} \subset \mathcal{V} \times \mathcal{V}$$

[2] In local coordinates and with $v = \frac{1}{(k+1)!} v^{i_1 \cdots i_{k+1}} \partial_{i_1} \wedge \cdots \wedge \partial_{i_{k+1}}$ the action of $\widehat{U}_1$ is given in the usual way, namely $\widehat{U}_1(v)(f_1 \otimes \cdots \otimes f_{k+1}) = \frac{1}{(k+1)!} v^{i_1 \cdots i_{k+1}} \partial_{i_1} f_1 \cdots \partial_{i_{k+1}} f_{k+1}$.

(C) For a graph Γ we denote its set of vertices and edges by $\mathcal{V}(\Gamma)$ and $\mathcal{E}(\Gamma)$, respectively.

(D) For all numbers $j \in \mathbb{N}$ we define the set $\mathrm{Start}(\Gamma, j)$ as the set of edges starting at the vertex labeled by j and the integer $\mathrm{start}(\Gamma, j)$ as the amount of such edges, i.e.

$$\mathrm{Start}(\Gamma, j) := \{e \in \mathcal{E}(\Gamma); e_1 = j\}, \qquad \mathrm{start}(\Gamma, j) := \#\mathrm{Start}(\Gamma, j)\,. \quad (31)$$

We emphasize the relation $\#\mathcal{E}(\Gamma) = \sum_{i \in \mathbb{N}} \mathrm{start}(\Gamma, i)$.

(E) We denote the set of graphs Γ with n vertices of first kind, m vertices of second kind, and $2n + m - 2 - \ell$ edges by $G^{\ell}_{n,\bar{m}}$, i.e.

$$G^{\ell}_{n,\bar{m}} := \{\Gamma\,;\ \#\mathcal{V}(\Gamma)_1 = n, \#\mathcal{V}(\Gamma)_2 = m, \#\mathcal{E}(\Gamma) = 2n + m - 2 - \ell\}\,. \quad (32)$$

To each graph $\gamma \in G^{\ell}_{n,\bar{m}}$ we associate a map U_Γ from $\otimes^n \mathfrak{T}_{\mathrm{poly}}$ to $\mathfrak{D}_{\mathrm{poly}}[1-n]$ such that exactly one graded component survives.

Namely, in the source U_Γ acts non-trivially on

$$(\otimes^n \mathfrak{T}_{\mathrm{poly}})^{(k_1 + \cdots + k_n)}$$

with $k_i := \mathrm{start}(i) - 1$, and has image in

$$\mathfrak{D}^{(m-1)}_{\mathrm{poly}} = \mathfrak{D}_{\mathrm{poly}}[1-n]^{(n+m-2)}.$$

The degree of the map U_Γ then is

$$\begin{aligned} |U_\Gamma| &= (n+m-2) - \textstyle\sum_{i=1}^n k_i = (n+m-2) - \sum_{i=1}^n (\mathrm{start}(i) - 1) \\ &= 2n + m - 2 + \#\mathcal{E}(\Gamma) = 2n + m - 2 + (2n + m - 2 - \ell) = \ell \quad (33) \end{aligned}$$

Remark 6.1. The graph Γ may give a contribution to a Taylor coefficient of an L_∞-morphism $U : \mathfrak{T}_{\mathrm{poly}} \to \mathfrak{D}_{\mathrm{poly}}$ only if $\Gamma \in G^0_{n,\bar{m}}$, i.e. Γ has exactly $2n + m - 2$ edges for some $m, n \in \mathbb{N}$.

Let $\Gamma \in G^0_{n,\bar{m}}$ be a graph. For $v_1 \otimes \cdots v_n \in \mathfrak{T}^{(k_1)}_{\mathrm{poly}} \otimes \cdots \otimes \mathfrak{T}^{(k_n)}_{\mathrm{poly}}$ with $k_i = \mathrm{start}(\Gamma, i) - 1$ and $f_1 \otimes \cdots \otimes f_m \in \otimes^m C^\infty$ we are going to define

$$U_\Gamma(v_1 \otimes \cdots \otimes v_n)(f_1 \otimes \cdots \otimes f_m)\,. \quad (34)$$

In particular, the values $\mathrm{start}(i)$ determine the degrees of the poly-vector fields v_i for which the value is non-zero with respect to the grading of $\mathfrak{X}(M)$.

Consider a map $I : \mathcal{E}(\Gamma) \to \{1, \ldots, D\}$ and associate to each vertex of Γ a function, namely

$$\begin{aligned} \mathcal{V}_2(\Gamma) \ni \bar{j} \ &\longmapsto\ \psi^I_{\bar{j}} := f_j \\ \mathcal{V}_1(\Gamma) \ni i \ &\longmapsto\ \psi^I_i := \left\langle \tilde{v}_i\,,\, dx^{I(i,1)} \otimes \cdots \otimes dx^{I(i,\mathrm{start}(i))} \right\rangle \end{aligned} \quad (35)$$

where we identify poly-vector fields and skew-symmetric tensor fields in the usual way

$$\mathfrak{T}^{(k)}_{\text{poly}} \ni v_i = x_1 \wedge \cdots \wedge x_{k+1} \\ \longleftrightarrow \ \tilde{v}_i = \sum_{\sigma \in \mathfrak{S}_{k+1}} (-)^{\sigma} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(k+1)} \in \left(\mathfrak{X}^{(1)}\right)^{\otimes(k+1)} . \tag{36}$$

In the next step we replace the function associated to each vertex by its partial derivatives in the following way

$$\mathcal{V}(\Gamma) \ni v \ \longmapsto \ \Big(\prod_{\mathcal{E}(\Gamma) \ni e = (\cdot\,, v)} \partial_{I(e)} \Big) \psi^I_v \ . \tag{37}$$

The value (34) is the defined as the sum over all maps I and the product over all vertices, i.e.

$$U_\Gamma(v_1 \otimes \cdots \otimes v_n)(f_1 \otimes \cdots \otimes f_m) = \sum_{I : \mathcal{E}(\Gamma) \to \{1, \ldots, D\}} \prod_{v \in \mathcal{V}(\Gamma)} \Big(\prod_{\mathcal{E}(\Gamma) \ni e = (\cdot\,, v)} \partial_{I(e)} \Big) \psi^I_v \, . \tag{38}$$

Remark 6.2. (1) The enumeration of the set

$$\text{Start}(i) = \big\{ (i,1), \ldots, (i, \text{start}(i)) \big\}$$

determines the sign of the value (38).

(2) If we permute the vertices in Γ by a permutation σ and call the resulting graph Γ^σ we have

$$U_{\Gamma^\sigma}(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) = U_\Gamma(v_1 \otimes \cdots \otimes v_n)$$

As mentioned in Remark 6.1 the collection of maps associated to the admissible graphs such that their degree is zero will be the starting point for the definition of the L_∞-morphism.

7. Configuration spaces and weights associated to graphs

The configuration spaces are given by

$$\text{Conf}_n := \big\{ (z_1, \ldots, z_n) \in \mathbb{C}^n ; \ z_i \neq z_j \big\}$$

and

$$\text{Conf}_{n,m} \\ := \big\{ (z_1, \ldots, z_n, \alpha_1, \ldots, \alpha_m) \in \mathbb{C}^n \times \mathbb{R}^m ; \ \alpha_i \neq \alpha_j, z_i \neq z_j, \text{Im}(z_j) > 0 \big\} \, .$$

There are subgroups of the Möbius transformations act on these spaces. Namely $G^3 = \{ z \mapsto \alpha z + b; \ \alpha \in \mathbb{R}^{>0}, b \in \mathbb{C} \}$ on Conf_n leaving ∞ invariant, and $G^2 = \{ z \mapsto \alpha z + \beta; \ \alpha, \beta \in \mathbb{R}, \alpha > 0 \}$ on $\text{Conf}_{n,m}$ leaving ∞ and the real axis invariant. The action is faithful so that the quotient spaces are manifolds of dimension 3 and 2 less:

$$\begin{aligned} C_{n,m} &:= \mathrm{Conf}_{n,m}/G^2, \qquad \dim C_{n,m} = 2n+m-2 \\ C_n &:= \mathrm{Conf}_n/G^3, \qquad \dim C_n = 2n-3 \end{aligned} \tag{39}$$

Consider the graph $\Gamma \in G^{\ell}_{n,\bar{m}}$ embedded in the upper half plane H provided with the hyperbolic metric. The embedding is in such a way that the edges are geodesics, i.e. the edge connecting two vertices (=points) is part of the circle centered on the real line connecting these two points. For two points $p, q \in H$ we define the angle via

$$\phi(p,q) := \measuredangle(e(p,\infty), e(p,q)) = arg\big(\frac{p-q}{p-\bar{q}}\big)\,.$$

This angle map yields an 1-form $d\phi$ on $C_{2,0}$. This is pulled back to $C_{n,\bar{m}}$ via the forgetful map $p_{i,j} : C_{n,m} \to C_{2,0}$, $p_{i.j}([z_1, \ldots, z_n, \alpha_1 \ldots, \alpha_m]) = [z_i, z_j]$. The result is a collection of 1-forms $d\phi_{i,j} = p^*_{i,j} d\phi$ on $C_{n,m}$.

The angle form of a graph is now the wedge product of all the 1-forms obtained by the pull backs via edges. We define the WEIGHT W_Γ associated to a graph Γ to be proportional to the angle form of the graph integrated over the (compactified) configuration space $\bar{C}_{n,m}$, i.e.

$$W_\Gamma = \frac{1}{\prod_{k=1}^n \mathrm{start}(k)!} \frac{1}{(2\pi)^{2n+m-2}} \int_{\bar{C}_{n,m}} \bigwedge_{e \in \mathcal{E}(\Gamma)} d\phi_e \tag{40}$$

Remark 7.1. The sign of the weight depends on

(1) the numbering of the set $\mathrm{Start}(\Gamma, i)$. A change of this numbering yields the same sign as in Remark 6.2(1).
(2) the numbering of the vertices. After interchanging vertex i and j, the sign is determined by the values of start, namely

$$(-)^{\mathrm{start}(i)\mathrm{start}(j) + \sum_{p=i+1}^{j-1} (\mathrm{start}(i)+\mathrm{start}(j))\mathrm{start}(p)}\,.$$

More general, if we permute the vertices by σ we get

$$W_{\Gamma^\sigma} = \epsilon(\sigma, \Gamma) W_\Gamma$$

where $\epsilon(\sigma, \Gamma)$ is the sign we get after reordering graded elements of degree $\mathrm{start}(1), \ldots, \mathrm{start}(n)$ which have been shuffled by σ. We may say that this sign does only depends on the local structure of the graph around the vertices with non-vanishing imaginary part.

8. The L_∞-morphism

We are now in the position to write down the candidate for the L_∞-morphism U. As written above it is enough to give the Taylor coefficients. We define

$$U_n := \sum_{m\geq 0} \sum_{\Gamma \in G^0_{n,\bar{m}}} W_\Gamma U_\Gamma . \tag{41}$$

Remark 8.1. (1) The sum is finite, because

- if we insert a poly-vector field in U_n the, in the second sum many summand vanish This is due to the fact that the amount and degree of the poly vectors have to match with the amount of vertices and edges of the graph. More precisely, the amount of edges starting from one vertex have to match with the degree of the corresponding entry. I.e. only the graphs with admissible local structure enter into the sum.
- if we evaluate the multi-differential operator on an ℓ-fold tensor product of functions, only the summand with $m = \ell$ enters into the first sum.

(2) The value (41) is independent of the numbering of the sets $\mathrm{Start}(\Gamma, i)$, because of Remark 6.2(1) and 7.1(1).

The map U_n is graded symmetric with respect to the grading in $\mathfrak{T}_{\mathrm{poly}}[1]$, because for a permutation σ we have

$$\begin{aligned}
U_n(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) &= \sum_{m\geq 0} \sum_{\Gamma \in G^0_{n,\bar{m}}} W_\Gamma U_\Gamma (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}) \\
&\overset{(\text{Remark } 6.2(2))}{=} \sum_{m\geq 0} \sum_{\Gamma \in G^0_{n,\bar{m}}} W_\Gamma U_{\Gamma^\sigma} (v_1 \otimes \cdots \otimes v_n) \\
&\overset{(\text{Remark } 7.1(2))}{=} \sum_{m\geq 0} \sum_{\Gamma \in G^0_{n,\bar{m}}} \epsilon(\sigma, \Gamma) W_{\Gamma^\sigma} U_{\Gamma^\tau} (v_1 \otimes \cdots \otimes v_n) \\
&= \epsilon(\sigma, \Gamma) \sum_{m\geq 0} \sum_{\Gamma \in G^0_{n,\bar{m}}} W_\Gamma U_\Gamma (v_1 \otimes \cdots \otimes v_n) \\
&= \epsilon(v_{\sigma(1)}, \ldots, v_{\sigma(n)}) U_n (v_1 \otimes \cdots \otimes v_n)
\end{aligned}$$

The sign $\epsilon(\sigma, \Gamma)$ is constant for all Γ which match with the entries in U_n, because of Remark 7.1(2). In the last line the degrees of the entries have to be taken in $\mathfrak{T}_{\mathrm{poly}}[1]$. There we have $|v_i| = \mathrm{start}(i) - 2$ such that the last equality holds.

Example 8.2. The set $G^0_{1,\bar{m}}$ consists of one single graph Γ^m given by $\mathcal{V}(\Gamma^m) = \{1, \bar{1}, \ldots, \bar{m}\}$ and $\mathcal{E}(\Gamma^m) = \{(1, \bar{1}), \ldots, (1, \bar{m})\}$. The weight associated to this graph is $W_{\Gamma^m} = \frac{1}{m!}$. With these ingredients we are able to state the action of U_1 on an element $v = \frac{1}{k!} v^{i_1 \ldots i_k} \partial_{\sigma(i_1)} \wedge \cdots \wedge \partial_{\sigma(i_k)} \in \mathfrak{T}^{(k-1)}_{\mathrm{poly}}$

evaluated on k functions f_i, c.f.(41).

$$\begin{aligned} U_1(v)(f_1,\dots,f_k) &= \sum_{m\geq 0}\sum_{\Gamma\in G^0_{1,\bar m}} W_\Gamma U_\Gamma(v)(f_1,\dots,f_k) \\ &= \frac{1}{k!}U_{\Gamma^k}(v)(f_1,\dots,f_k) \\ &= \frac{1}{k!}\sum_{I:\{\bar 1,\dots,\bar k\}\to\{1,\dots,D\}} v^{I(\bar 1)\dots I(\bar k)}\prod_{\bar i=\bar 1}^{\bar k}\partial_{I(\bar i)}f_{\bar i} \\ &= \frac{1}{k!}v^{i_1\dots i_k}\partial_{i_1}f_1\cdots\partial_{i_k}f_k\,. \end{aligned}$$

Here we used

$$\begin{aligned} \psi_1^I &= \left\langle \tilde v\,,\, dx^{I(\bar 1)}\otimes\cdots\otimes dx^{I(\bar k)}\right\rangle \\ &= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} v^{\sigma(i_1)\dots\sigma(i_k)}\left\langle \partial_{\sigma(i_1)}\otimes\cdots\otimes\partial_{\sigma(i_k)}, dx^{I(\bar 1)}\otimes\cdots\otimes dx^{I(\bar k)}\right\rangle \\ &= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} v^{\sigma(i_1)\dots\sigma(i_k)}\delta^{I(\bar 1)}_{\sigma(i_1)}\cdots\delta^{I(\bar k)}_{\sigma(i_k)} \\ &= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} v^{I(\bar 1)\dots I(\bar k)} \\ &= v^{I(\bar 1)\dots I(\bar k)} \end{aligned}$$

with

$$\begin{aligned} \tilde v &= \frac{1}{k!}v^{i_1\dots i_k}\sum_{\sigma\in\mathfrak{S}_k}(-)^\sigma\partial_{\sigma(i_1)}\otimes\cdots\otimes\partial_{\sigma(i_k)} \\ &= \frac{1}{k!}\sum_{\sigma\in\mathfrak{S}_k} v^{\sigma(i_1)\dots\sigma(i_k)}\partial_{\sigma(i_1)}\otimes\cdots\otimes\partial_{\sigma(i_k)}\,. \end{aligned}$$

9. Poisson structures and $*$-products

In Section 5 we gave the two Maurer-Cartan sets for the poly-vector fields and poly-differential operators. An L_∞-morphism yields a one to one correspondence between these sets, given by (26).

In terms of the maps constructed in the proceeding section, we can associate to every Poisson structure π a $*$-product. In the formula we add the 0^{th} order term given by the multiplication.

$$f\star g - fg := \sum_n \frac{\hbar^n}{n!}U_n(\pi^n)(f\otimes g) = \sum_n\frac{\hbar^n}{n!}\sum_{\Gamma\in G^0_{n,\bar 2}} W_\Gamma U_\Gamma(f\otimes g)\,. \tag{42}$$

In particular, the first order terms are

$$f \star g = fg + \hbar\{f, g\} + o(\hbar^2) \tag{43}$$

where the bracket is the Poisson bracket induced by π, i.e

$$\{f, g\} = \frac{1}{2}\pi^{ij}\partial_i f \partial_j g\,.$$

References

[1] Enrico Arbarello. Introduction to Kontsevich's result on deformation quantization of Poisson structures. In *Algebraic geometry seminars. 1998–1999. Papers from the seminars held at the Scuola Normale Superiore, Pisa, 1998–1999. (Seminari di geometria algebrica. 1998–1999)*, page 287 p. Pisa: Scuola Normale Superiore., 1999.

[2] D. Arnal, D. Manchon, and M. Masmoudi. Choix des signes pour la formalité de M. Kontsevich. `math.QA/0003003`, 2000.

[3] Paolo de Bartolomeis. GBV algebras, formality theorems, and Frobenius manifolds. In *Algebraic geometry seminars. 1998–1999. Papers from the seminars held at the Scuola Normale Superiore, Pisa, 1998–1999. (Seminari di geometria algebrica. 1998–1999)*, page 287 p. Pisa: Scuola Normale Superiore., 1999.

[4] Alberto Canonaco. L_∞-algebras and quasi-isomorphisms. In *Algebraic geometry seminars. 1998–1999. Papers from the seminars held at the Scuola Normale Superiore, Pisa, 1998–1999. (Seminari di geometria algebrica. 1998–1999)*, page 287 p. Pisa: Scuola Normale Superiore., 1999.

[5] Alberto S. Cattaneo. Formality and star products. `math.QA/0403135v1`, 2004.

[6] Maxim Kontsevich. Deformation quantization of Poisson manifolds, I. `q-alg/9709040`, 1997.